

Rate of Consolidation: Terzaghi's Theory of 1-D Consolidation

Lecture No. 14
November 05, 2002
November 07, 2002

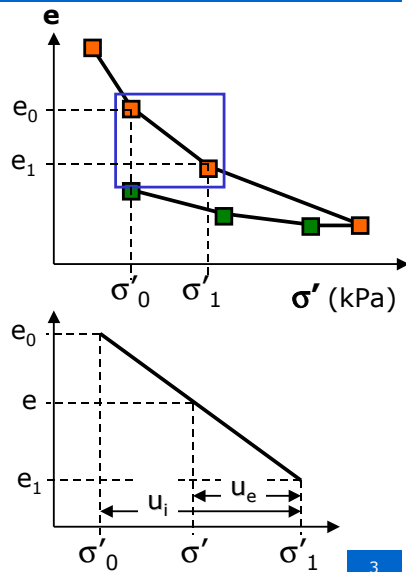
Degree of Consolidation (Continued..)

- If we assume the $e-\sigma'$ curve to be **linear** over the range of stresses in the field (see figure), the degree of consolidation can be expressed in terms of effective stresses as:

$$U_z = \frac{\sigma' - \sigma'_0}{\sigma'_1 - \sigma'_0}$$

where σ' is the effective stress in the soil at an instant of time t and is given by:

$$\sigma' = \sigma'_0 + \Delta\sigma' = \sigma'_0 + \Delta\sigma - u_e$$



Degree of Consolidation

- For an element of soil at a **depth z** , the progress of the consolidation process under a particular total stress increment can be measured in terms of the **void ratio** at an instant of **time t** as:

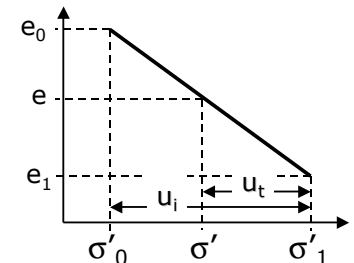
$$U_z = \frac{e_0 - e}{e_0 - e_1}$$

e_0 - void ratio before the start of consolidation
 e_1 - void ratio at the end of consolidation
 e - void ratio at time t after the start of consolidation

- U_z is defined as the **degree of consolidation** at a particular instant of **time t** at **depth z** .
- U_z takes a value between 0 and 1. At the **start** of consolidation $U_z = 0$ and at the **end** of consolidation, $U_z = 1$.

Degree of Consolidation (Continued..)

- Since the excess pore pressure is zero at the end of consolidation, the effective stress at the end of consolidation can be expressed as:



$$\sigma'_1 = \sigma'_0 + \Delta\sigma' = \sigma'_0 + \Delta\sigma \quad [\because \Delta\sigma' = \Delta\sigma \text{ at the end of consolidation}]$$

- Therefore, the degree of consolidation can be expressed in terms of changes in excess pore pressure as:

$$U_z = \frac{\sigma'_0 + \Delta\sigma - u_e - \sigma'_0}{\sigma'_0 + \Delta\sigma - \sigma'_0} = \frac{\Delta\sigma - u_e}{\Delta\sigma} = 1 - \frac{u_e}{u_i}$$

[$\because u_i = \Delta\sigma$ at the start of the consolidation]

Terzaghi's Theory of 1-D Consolidation

- Terzaghi's theory of 1-D consolidation makes the following simplifying assumptions:
 - The soil is homogeneous.
 - The soil is fully saturated.
 - The solid particles and the pore water are incompressible.
 - The flow of water and compression of soil are one-dimensional (vertical).
 - Strains are small.
 - Darcy's law is valid at all hydraulic gradients.
 - The coefficient of permeability and the coefficient of volume compressibility remain constant throughout the consolidation process.
 - There is a unique relationship, independent of time, between void ratio and effective stress.

5

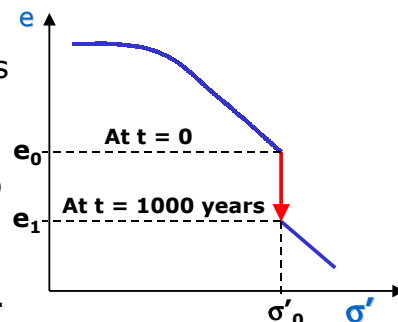
Terzaghi's Theory (Continued..)

- Assumptions 1 to 5 are reasonable and therefore, pose no difficulties in applying Terzaghi's theory to practical problems.
- At very low hydraulic gradients, there is evidence that pore water flow doesn't take place according to Darcy's law as stated in Assumption No. 6. However, for most fine-grained soils, the hydraulic gradient is sufficiently high and therefore, this assumption is OK.
- We have seen that the coefficient of permeability (k) and the coefficient volume compressibility (m_v) decreases with increasing effective stress. However, for small stress increments, Assumption No. 7 is reasonable.

6

Terzaghi's Theory (Continued..)

- The main limitation of Terzaghi's theory originates from Assumption No. 8.
- Experimental results have shown that the relationship between the **void ratio** and **effective stress** is **not independent of time**.
- Most fine-grained soils undergo a decrease in void ratio with time (called **secondary consolidation** or **creep**) at **constant** effective stress as shown in the figure above.
- Therefore, Terzaghi's theory is good only for the estimation of the rate of **primary** consolidation.



7

Terzaghi's Theory (Continued..)

- Terzaghi's theory relates the following three quantities:
 - The **excess** pore water pressure (previously denoted by Δu but from now on it will be denoted by u_e)
 - The depth (z) below the top of the clay layer.
 - The time (t) measured from the start of the consolidation, i.e. the instant at which the total stress increment was applied.
- The governing differential equation of consolidation according to Terzaghi's theory is:

$$\frac{\partial u_e}{\partial t} = c_v \frac{\partial^2 u_e}{\partial z^2}$$
 where c_v is called the **coefficient of consolidation** and is given by:

$$c_v = k / (m_v \gamma_w) \text{ [units } m^2/\text{day or } m^2/\text{year}]$$
- Since k and m_v are constant, c_v is also constant.

8

Solution of the Consolidation Equation

- The partial differential equation of consolidation can be solved by first rewriting it in a variable separable form and then making use of Fourier Series to solve it for particular boundary conditions.
- Details of this solution are out of scope of this course but you can find the details in any book on advanced soil mechanics.
- The solution is written in terms of excess pore water pressure as:

$$u_e = \sum_{m=0}^{m=\infty} \frac{2u_i}{M} \sin\left(\frac{Mz}{d}\right) e^{-M^2 T_v}$$

where u_i is initial excess pore pressure and d is the length of the longest drainage path and

$$M = \frac{\pi}{2}(2m+1)$$

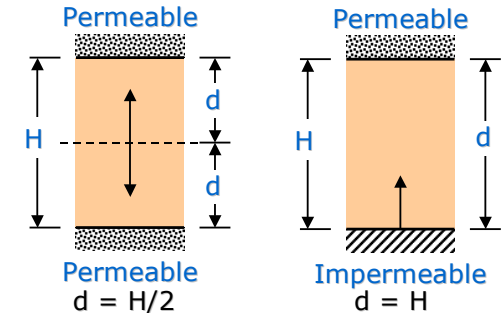
$$T_v = c_v t / d^2$$

- T_v is a dimensionless number called the **time factor**.

9

Key Features of the Solution

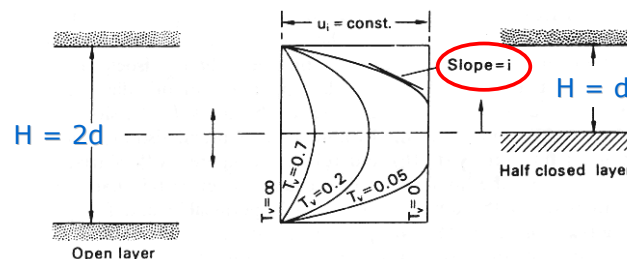
- The length of the longest drainage path is shown in the figure below:
- A layer for which both the upper and the lower boundaries are permeable or free-draining is called an **open layer**.
- A layer for which only one boundary is permeable is called a **half-closed layer**.
- Since $t \propto d^2$, a half-closed layer requires **four times as much time** to consolidate than an open layer of same thickness H .



10

Key Features (Continued..)

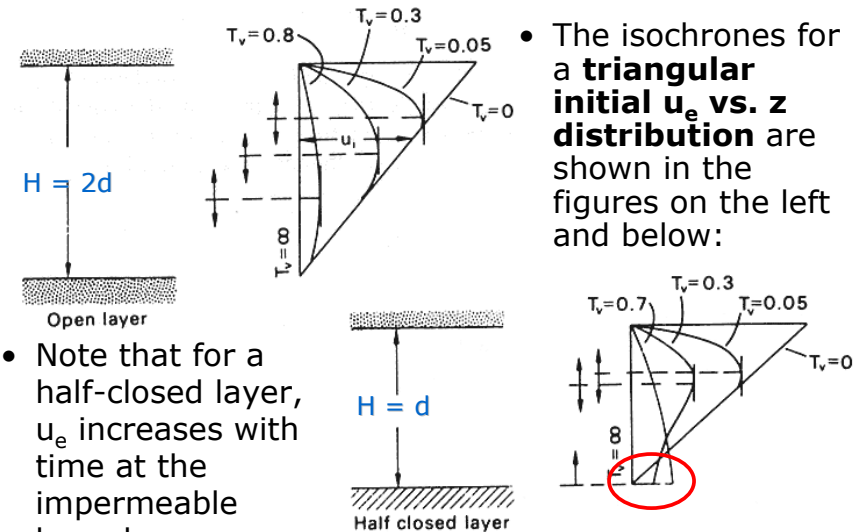
- The solution can be used to view the progress of consolidation by plotting a series of curves of u_e vs. z at different values of t .
- Such curves are called **isochrones** and their form depends on the drainage boundary conditions and the initial distribution of u_e vs. z .
- The isochrones for an initially constant u_e vs. z distribution are shown in the figure below.



11

Key Features (Continued..)

- The isochrones for a **triangular initial u_e vs. z distribution** are shown in the figures on the left and below:
- Note that for a half-closed layer, u_e increases with time at the impermeable boundary.



12

Key Features (Continued..)

- The solution can be written in terms of degree of consolidation U_z at a time t and at any depth z by substituting the solution in terms of u_e into the equation for U_z (given on page 4):

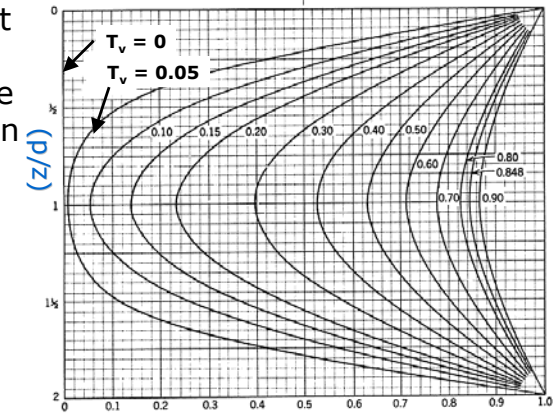
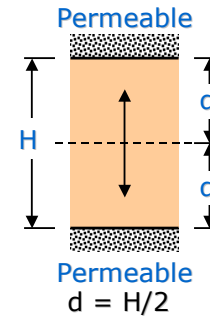
$$U_z = 1 - \frac{u_e}{u_i} = 1 - \sum_{m=0}^{\infty} \frac{2}{M} \sin\left(\frac{Mz}{d}\right) e^{-M^2 T_v}$$

- Therefore, the solution can be used to calculate degree of consolidation U_z at any instant of time and at any depth z .
- Often, a parametric chart of (z/d) vs. U_z showing isochrones at various T_v values is used for a given boundary condition.

13

Chart of (z/d) vs. U_z

- Figure on the right shows a chart of (z/d) vs. U_z for the boundary condition shown below.

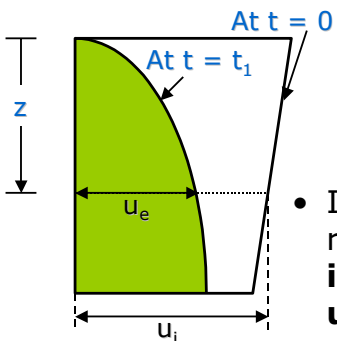


- You can get U_z at any depth for a given T_v or time required to reach a given U_z at a given depth.

14

Average Degree of Consolidation

- For practical problems, it is much more useful to calculate an average degree of consolidation over the entire depth of the clay layer.
- The average degree of consolidation (U_{avg}) for a general distribution of u_i vs. z is defined as:



$$U_{avg} = 1 - \frac{\int_0^H u_e dz}{\int_0^H u_i dz}$$

- In simple terms, U_{avg} is 1 minus the ratio of the **area under the isochrone at $t = t_1$** over the **area under the initial u_i vs. z distribution**.

15

Average Degree of Consolidation (Continued..)

- For an initially constant u_i distribution, the average degree of consolidation can be written as:

$$U_{avg} = 1 - \frac{\int_0^H u_e dz}{u_i} = 1 - \sum_{m=0}^{\infty} \frac{2}{M^2} e^{-M^2 T_v}$$

- The above equation can be represented almost exactly by the following empirical equations:

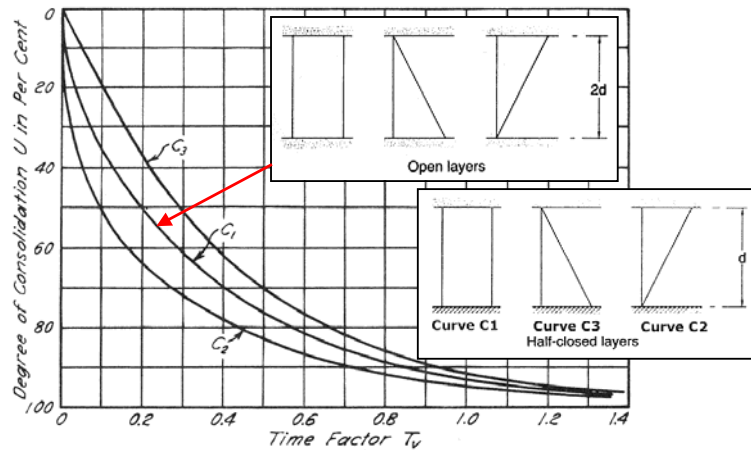
$$\text{for } U_{avg} \leq 0.60, T_v = \frac{\pi}{4} U_{avg}^2$$

$$\text{for } U_{avg} > 0.60, T_v = -0.933 \log(1 - U_{avg}) - 0.085$$

16

Average Degree of Consolidation (Continued..)

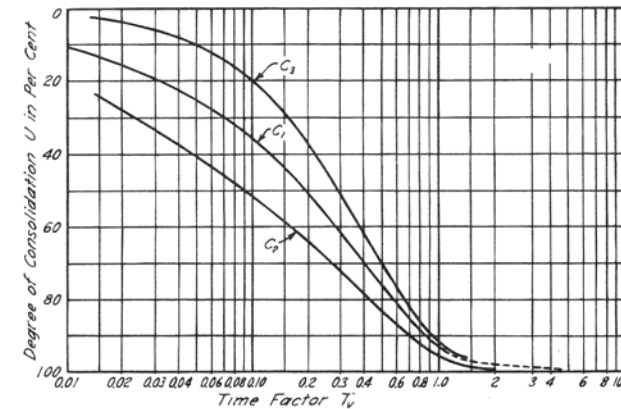
- The exact equation for average degree of consolidation (given on page 16) for an open layer can be represented by curve C1 in the figure below:



17

Average Degree of Consolidation (Continued..)

- The curves on page 17 are not very accurate at earlier stages of consolidation (small values of T_v).
- This problem can be resolved if T_v is plotted on a logarithmic scale as shown in the figure below:



18

Average Degree of Consolidation (Continued..)

- Since the settlement of a clay layer depends on the cumulative gain in effective stress, the average degree of consolidation can be written in terms of settlement as:

$$U_{avg} = \frac{s_t}{s_{ult}}$$

s_t – settlement of the clay layer at time t
 s_{ult} – ultimate settlement of the clay layer (at the end of consolidation)

- In the above equation, s_{ult} can be calculated using the Compression and Expansion indices (C_c and C_e) as described in Lecture No. 13.
- Therefore, if the average degree of consolidation is known, the settlement at any time t can be easily estimated.

19

Rate of Consolidation – Example #1

- A soft clay layer **2.5 m** thick is sandwiched between two sand layers. The initial total vertical stress at the center of the clay layer is **200 kPa** and the pore water pressure is **100 kPa**. The increase in vertical stress at the center of the clay layer from a building foundation is **100 kPa**. What will be the effective vertical stress and excess pore water pressure at the center of the clay layer when the degree of consolidation at the center of the clay layer is **60%**?

[This example will be solved during the class.]

20

Rate of Consolidation – Example #2

- An undisturbed sample, **75 mm** in diameter and **20 mm** high, taken from a **10 m** thick clay layer, was tested in an oedometer with drainage at both the upper and the lower boundaries. It took the sample **15 minutes** to settle by **3.5 mm**. The ultimate settlement of the sample at the end of consolidation was **7.0 mm**.
 - If the clay layer in the field has the same drainage conditions as the laboratory sample, calculate the time the clay layer will take to achieve **50%** and **90%** average degrees of consolidation.
 - If the clay layer in the field had drainage only from the top boundary, how long will it take to achieve **50%** average degree of consolidation?

[This example will be solved during the class.]

21

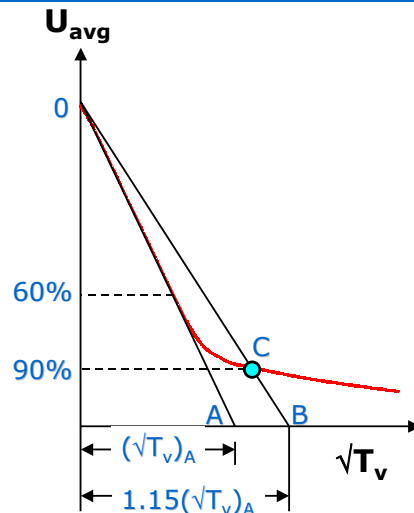
Determination of Coefficient of Consolidation

- There are two popular methods that can be used to determine the coefficient of consolidation (c_v) of a clay layer:
 - Root Time Method
 - Log Time Method
- The Root Time Method was proposed by D.W. Taylor in 1942 and the Log Time Method was proposed by Casagrande and Fadum in 1940.
- The **Root Time Method** utilizes the early settlement response which theoretically should appear as a **straight line** in a plot of dial gauge reading (settlement) vs. square root of time.

22

Root Time Method

- Referring to the approximate relationship between U_{avg} and T_v given on page 16, the theoretical distribution of U_{avg} with $\sqrt{T_v}$ should be linear up to $U_{avg} = 60\%$.
- Let us arbitrarily choose a point C on this theoretical curve, as shown in the figure on the right and assume that this point corresponds to $U_{avg} = 90\%$.



23

Root Time Method (Continued..)

- At $U_{avg} = 90\%$, the theoretical value of $T_v = 0.848$ (using the equations on page 16 or using the charts on pages 17 & 18).
- If point C were to lie on a straight line, the theoretical relationship between U_{avg} and T_v would be:

$$U_{avg} = 0.9 = 0.98 \cdot \sqrt{0.848} = 0.98 \sqrt{T_v}$$

- From the equations on page 16, the relationship between U_{avg} and T_v is given by:

$$T_v = \frac{\pi}{4} U_{avg}^2 \text{ for } U_{avg} \leq 0.60 \quad \text{or} \quad U_{avg} = 1.13 \sqrt{T_v}$$

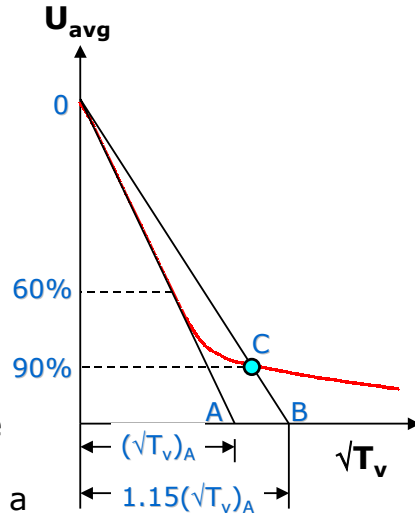
24

Root Time Method (Continued..)

- The early time response is denoted by the line OA in the figure.
- Therefore, the ratio of the abscissa of a point on line OCB to that of a point on line OA is:

$$\frac{1.13\sqrt{T_v}}{0.98\sqrt{T_v}} = 1.15$$

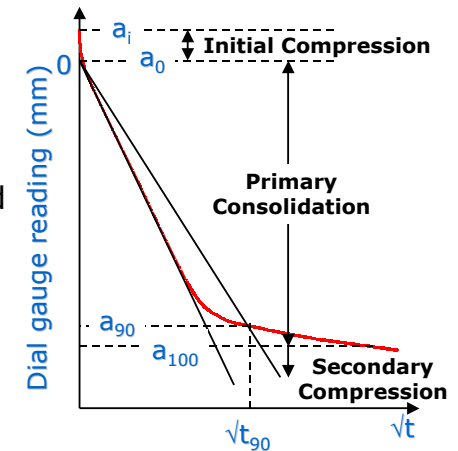
- Therefore, line OCB can be obtained from line OA by multiplying the abscissa of a point on line OA by 1.15.



25

Root Time Method (Continued..)

- The experimental curve of dial gauge reading (mm) vs. \sqrt{t} usually consists of a short curve representing initial compression, a linear part and a second curve as shown in the figure on the right.
- Point representing $U_{avg} = 0$ is obtained by extending the linear part of the curve backwards.



26

Root Time Method (Continued..)

- A straight line OE is then drawn that has abscissae 1.15 times the corresponding abscissae on the linear part of the curve.
- This line intersects the curve at point E that denotes $U_{avg} = 90\%$.
- The value of time at this point is t_{90} .
- The coefficient of consolidation can now be calculated as:

$$c_v = \frac{(T_v)_{90} d^2}{t_{90}} = \frac{0.848 d^2}{t_{90}}$$

27

Determination of c_v – An Example

- Following readings were taken for an increment of vertical stress of **50 kPa** in an oedometer test on a saturated clay sample of diameter **75 mm** and thickness **20 mm**. Drainage was permitted from both the top and the bottom boundaries. Determine the coefficient of consolidation (c_v) using the root time method.

Time (min)	0.25	1	2.25	4	9	16	25	36	24 hours
ΔH (mm)	0.12	0.23	0.33	0.43	0.59	0.68	0.74	0.76	0.89

[This example will be solved during the class.]

28